Molecular Crystals and Liquid Crystals
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gmcl16

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Version of record first published: 28 Mar 2007.

To cite this article: S. Holmström & S. T. Lagerwall (1977): Shear Flow in the Presence of Electric Fields: A Method for Separate Determination of Elastic and Viscous Constants of Nematics, Molecular Crystals and Liquid Crystals, 38:1, 141-153

To link to this article: http://dx.doi.org/10.1080/15421407708084381

Mol. Cryst. Liq. Cryst., 1977, Vol. 38, pp. 141–153

Shear Flow in the Presence of Electric Fields†

A Method for Separate Determination of Elastic and Viscous Constants of Nematics

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(*Received October 18, 1976*)

We have investigated a method for separate determination of elastic and viscous constants for nematics in a rotational shear flow experiment. By inducing a shear flow perpendicular to the initial molecular alignment and simultaneously applying an electric a.c. field at right angles to the flow it is possible to determine not only k_{11}/k_{33}, $(\gamma_2 - \gamma_1)/k_{33}$ and γ_2/γ_1, but split the combinations into k_{11}, k_{33}, γ_1, γ_2 and η_2 (Miesowicz). Although the flow solutions tend to be instable in the case of negative dielectric anisotropy, the required electric fields can be shown to lie far below the instability thresholds. Thus the method can be quite generally applied to all nematics, of both $\varepsilon_a > 0$ and $\varepsilon_a < 0$. Together with the widely used method for determining k_{22} (unwinding) it offers a rapid and accurate access to the most important elastic and viscous constants of nematic liquids.

1 INTRODUCTION

Methods for determining elastic and viscous constants of nematics have been available for some time. The elastic constants are usually determined by using thin uniformly aligned nematic layers in order to achieve simple and well defined geometries. The layers are then deformed by induced magnetic

† This paper was presented at The Sixth International Liquid Crystal Conference at Kent State University, August 22–27, 1976.

fields[1] (sometimes electric fields) and what is measured are combinations of the type K/χ_a between the elastic constants K and the magnetic anisotropy χ_a.

For measuring viscous constants different techniques have been used: shear flow under a strong orienting field,[2] capillary flow,[3,4] ultrasonic attenuation,[5,6] rotating magnetic fields,[7,8] pulsed fields,[9,10] light scattering.[11,12] Recently a new optical method was introduced by Wahl and Fischer.[13] It is a simple shear flow experiment without external fields. A flow is induced in a nematic between two parallell circular glass plates with homeotropic boundary conditions by rotating the plates relative to each other. By shining monochromatic light normal onto the sample which is placed between crossed polarizers one can measure the average optical anisotropy as a function of the shear velocity. In this way it is possible to determine combinations of elastic and viscous parameters. Waltermann and Fischer[14] have modified this method by applying magnetic fields perpendicular and parallell to the sample. In so doing they were able to make separate measurements of some of the parameters.

McQueen *et al*[15] have modified the Wahl and Fischer method by applying an electric d.c. field normal to the sample. However, this method is rather unreliable because complications can arise. Firstly instabilities can easily occur and secondly it is not easy to control the consequences of charge injection into the nematic.

In this paper we investigate a further modification which uses an electric a.c. field. This method has been proposed by Fischer, Wahl and Waltermann.[16] By choosing the field strength and frequency properly it is easy to avoid the abovementioned difficulties. The analysis that follows starts with the same assumption as Ref. 13, i.e. we use the formula $m\lambda_0/d = \langle n_e - n_0 \rangle$, where m is the number of a certain ring in the conoscopic pattern, λ_0 is the incident wave length, d is the thickness of the sample and $\langle n_e - n_0 \rangle$ is the average optic anisotropy. We make a rather detailed analysis starting from the general equations formulated by Ericksen,[17–19] Leslie[20,21] and Parodi[22] (ELP). Our approach differs in some details from that in Refs. 13 and 14, however the final result is the same.

2 THE HYDRODYNAMIC EQUATIONS FOR A NEMATIC

As a starting point we write down the equations describing the motion of a nematic in an electromagnetic field. We use the ELP formulation. The equations can be separated into two groups. One that gives the conservation laws and one that gives a set of constitutive equations.

First of all, however, it is proper to introduce a few notations and definitions:

i) Vectors and tensors are given by their Cartesian components a_i and A_{ij} respectively. Where combinations of vectors and tensors occur, repeated indices means that the expression is to be summed over these indices.

ii) The following symbols for derivatives are used

$$\partial_i \equiv \frac{\partial}{\partial x_i}, \qquad \partial_t \equiv \frac{\partial}{\partial t}, \qquad D_t \equiv \partial_t + v_i \partial_i$$

where v_i is the flow velocity of the fluid.

iii) The vector n_i is the director, i.e. the preferred direction of the nematic, and the scalar ρ is the mass density. We specify $n_i n_i = 1$.

Conservation laws

With these preliminaries we can write down the conservation laws. First of all we have the conservation of mass, i.e. the equation of continuity:

$$\partial_t \rho + \partial_i(\rho v_i) = 0 \tag{1}$$

As in ordinary hydrodynamics we have an equation that conserves linear momentum:

$$\rho D_t v_i = f_i + \partial_j \sigma_{ji} \tag{2}$$

Here f_i is a body force that in our treatment is limited to gravitational forces and therefore not taken into account any further. All electromagnetic forces are included into the stress tensor σ_{ij} via Maxwell's stress tensor. Since we have external eletromagnetic fields and since charges can occur in the nematic we must have an equation that conserves charge:

$$\partial_t q + \partial_i J_i = 0 \tag{3}$$

In this equation q is the charge density and J_i is a combination of conductive and convective currents.

The outstanding feature that distinguishes an anisotropic liquid like a nematic from an ordinary liquid, is the existence of the director. In a complete description we must have an equation that describes the motion of the director. This is:

$$I D_t D_t n_i = g_i + \partial_j \Pi_{ji} \tag{4}$$

The quantity I is the moment of inertia of the molecule, g_i includes electromagnetic and other effects and Π_{ij} is a stress tensor.

Constitutive equations

The list of constitutive equations will be long because the nematic liquid is not a simple system. We start with the stress tensors σ_{ij} and Π_{ij}:

$$\sigma_{ij} = -p\delta_{ij} - \Pi_{ik}\partial_j n_k + \hat{\sigma}_{ij} + \sigma_{ij}^{em} \tag{5}$$

$$\Pi_{ij} = \frac{\partial F}{\partial \partial_i n_j} \tag{6}$$

The entities occuring in these expressions are defined as follows (p is the ordinary hydrostatic pressure)

$$\hat{\sigma}_{ij} = \alpha_1 n_k n_p A_{kp} n_i n_j + \alpha_2 n_i N_j + \alpha_3 n_j N_i + \alpha_4 A_{ij} + \alpha_5 n_i n_k A_{kj} + \alpha_6 n_j n_k A_{ki} \tag{7}$$

is the viscous stress tensor, $\alpha_1, \ldots, \alpha_6$ are viscosity coefficients and $A_{ij} = (\partial_j v_i + \partial_i v_j)/2$.

$$\sigma_{ij}^{em} = \{D_i E_j + D_j E_i + B_i H_j + B_j H_i - (D_k E_k + B_k H_k)\delta_{ij}\}/2 \tag{8}$$

is Maxwell's stress tensor and

$$F = \{k_{11}\partial_i n_i \partial_j n_j + k_{22}(\varepsilon_{ijk} n_i \partial_j n_k)^2 + k_{33} n_i n_j \partial_i n_k \partial_j n_k - (D_i E_i + B_i H_i)\}/2 \tag{9}$$

is the elastic plus electromagnetic free energy and k_{11}, k_{22}, k_{33}, are Frank's[23] elasticity coefficients related to splay, twist and bend respectively. The fields are connected via

$$D_i = \varepsilon_{ij} E_j \quad \text{and} \quad B_i = \mu_{ij} H_j \tag{10}$$

where $\varepsilon_{ij} = \varepsilon_\perp \delta_{ij} + \varepsilon_a n_i n_j$, $\varepsilon_a = \varepsilon_\parallel - \varepsilon_\perp$, and $\mu_{ij} = \mu_\perp \delta_{ij} + \mu_a n_i n_j$, $\mu_a = \mu_\parallel - \mu_\perp$. Here $\parallel$ denotes parallell and $\perp$ perpendicular to the molecule. The vector g_i is given by

$$g_i = \gamma n_i - \frac{\partial F}{\partial n_i} + \hat{g}_i \tag{11}$$

where γ is a Lagrangian multiplier determined by the condition $n_i n_i = 1$ and $\hat{g}_i$ is defined by

$$\hat{g}_i = \gamma_1 N_i + \gamma_2 n_j A_{ij} \tag{12}$$

The coefficients γ_1 and γ_2 are two more viscosity coefficients (related to rotation) and N_i is the local angular velocity of the fluid given by

$$N_i = D_t n_i + \omega_{ki} n_k \tag{13}$$

with

$$\omega_{ij} = (\partial_j v_i - \partial_i v_j)/2.$$

The field D_i and q are connected via Poisson's equation

$$\partial_i D_i = q \tag{14}$$

The last constitutive equation befines the current J_i:

$$J_i = K_{ij} E_j + q v_i \tag{15}$$

where $K_{ij} = K_\perp \delta_{ij} + K_a n_i n_j$, $K_a = K_\| - K_\perp$ is the conductivity tênsor. Finally it is assumed that the liquid is incompressible i.e.

$$\partial_i v_i = 0 \tag{16}$$

3 APPLICATION TO THE PRESENT PROBLEM

In the introduction the experiment, to which the equations above are to be applied, is described. In this experiment the liquid is sheared so that it receives a cylindrical symmetry. However, since the shear velocities are small and since the radii of the abovementioned rings are very large compared to the molecular dimensions, we can approximate the circular motion with a linear motion. The actual geometry used is shown in Figure 1.

The use of an a.c. field is of experimental importance as pointed out in the introduction. A d.c. field may cause injection of charges into the sample and a voltage drop at the plates which we know very little about. It is also well known that d.c. fields and a.c. fields of low frequency may cause instabilities in the sample and give rise to an entirely new situation. Since we seek stationary solutions the a.c. frequency must be so high that the molecules will be unable to oscillate with field, i.e. their relaxation time must be much larger than ν^{-1}.

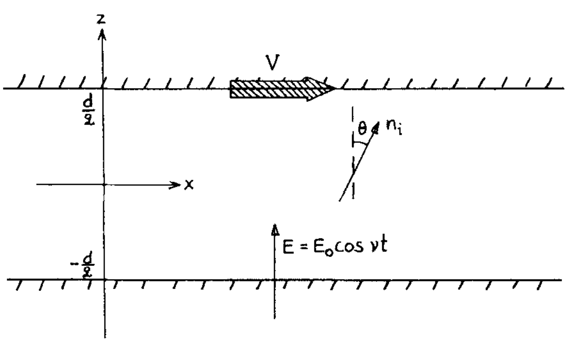

FIGURE 1 The distance between the plates is d and the upper plate is rotated with an angular velocity ω which gives a shear velocity $V = R\omega$, where R is the radius of a ring in the conoscopic picture. The only external field is an electric a.c. field in the z-direction. We approximate the rotation with a linear motion and constrain the director n_i to lie in the x–z plane with a tilt angle θ relative to the z axis.

If these conditions are fullfilled and if the voltage is not too large we are in the so called stable dielectric regime. Typical frequencies are of the order a few hundred Hz and typical voltages are below 100 V. However, as will be shown later, in our problem we must not use voltages much larger than 1 V. The field that the molecules feel will be E_{rms}, in our case $E_0/\sqrt{2}$.

As stated above we seek stationary i.e. time independent solutions, therefore all entities defined in §2 will be time independent. Also, since the liquid now is a dielectric, the conductivity tensor will be zero i.e. $K_{ij} = 0$. Furthermore, because of the symmetry, all entities will be only z dependent.

With all this in mind we now are in a position where we explicitly can write down all constitutive equations. We will just list them without further comments, since it is a straightforward matter to derive them from the general formulae (5)–(16). All entities not stated are zero.

$$n_x = \sin\theta, \qquad n_2 = \cos\theta, \qquad \theta = \theta(z), \qquad \frac{d\theta}{dz} = \theta', \qquad \frac{d^2\theta}{dz^2} = \theta'' \tag{17}$$

$$v_x = u(z), \qquad \frac{du}{dz} = u' \tag{18}$$

$$E_z = E_{rms}, \qquad D_x = \varepsilon_a E_{rms}\sin\theta\cos\theta, \qquad D_z = (\varepsilon_\perp + \varepsilon_a\cos^2\theta)E_{rms} \tag{19}$$

$$J_x = qu \tag{20}$$

$$A_{xz} = A_{zx} = u'/2 \tag{21}$$

$$\omega_{xz} = -\omega_{zx} = u'/2 \tag{22}$$

$$N_x = -u'\cos\theta/2, \qquad N_z = u'\sin\theta/2 \tag{23}$$

$$2F = k_{11}(\partial_z n_z)^2 + k_{33}n_z^2\{(\partial_z n_x)^2 + (\partial_z n_z)^2\} - (\varepsilon_\perp + \varepsilon_a n_z^2)E_{rms}^2 \tag{24}$$

$$\left.\begin{aligned}\sigma_{xx}^{em} &= \sigma_{yy}^{em} = -\sigma_{zz}^{em} = -(\varepsilon_\perp + \varepsilon_a\cos^2\theta)E_{rms}^2\\ \sigma_{xz}^{em} &= \sigma_{zx}^{em} = \varepsilon_a E_{rms}^2\sin\theta\cos\theta/2\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}2\hat{\sigma}_{xx} &= (2\alpha_1\sin^3\theta\cos\theta - (\alpha_2 + \alpha_3 - \alpha_5 - \alpha_6)\cos\theta\sin\theta)u'\\ 2\hat{\sigma}_{xz} &= (2\alpha_1\sin^2\theta\cos^2\theta + (\alpha_2 + \alpha_5)\sin^2\theta + (\alpha_6 - \alpha_3)\cos^2\theta + \alpha_4)u'\\ 2\hat{\sigma}_{zx} &= (2\alpha_1\sin^2\theta\cos^2\theta + (\alpha_5 - \alpha_2)\cos^2\theta + (\alpha_3 + \alpha_6)\sin^2\theta + \alpha_4)u'\\ 2\hat{\sigma}_{zz} &= (2\alpha_1\sin\theta\cos^3\theta + (\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6)\cos\theta\sin\theta)u'\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}\hat{g}_x &= -(\gamma_1 - \gamma_2)u'\cos\theta/2\\ \hat{g}_z &= (\gamma_1 + \gamma_2)u'\sin\theta/2\end{aligned}\right\} \tag{27}$$

$$\frac{\partial F}{\partial n_z} = k_{33}\theta'^2\cos\theta - \varepsilon_a E_{rms}^2\cos\theta \tag{28}$$

$$\Pi_{zx} = k_{33}\theta'\cos^3\theta, \qquad \Pi_{zz} = -(k_{11} + k_{33}\cos^2\theta)u'\sin\theta \tag{29}$$

$$\left.\begin{aligned}\partial_z \Pi_{zx} &= k_{33}\theta'' \cos^3\theta - 3k_{33}\theta'^2 \sin\theta\cos^2\theta \\ \partial_z \Pi_{zz} &= -\theta''(k_{11} + k_{33}\cos^2\theta)\sin\theta \\ &\quad - \theta'^2\{k_{11} + k_{33}(\cos^2\theta - 2\sin^2\theta)\}\cos\theta\end{aligned}\right\} \tag{30}$$

$$\Pi_{zk}\,\partial_z n_k = (k_{11}\sin^2\theta + k_{33}\cos^2\theta)\theta'^2 \tag{31}$$

From Eqs. (2) and (4) we get four equations

$$\partial_z(\hat{\sigma}_{zx} + \sigma_{zx}^{em}) = 0 \tag{32}$$

$$\partial_z(-p - \Pi_{zk}\,\partial_z n_k + \hat{\sigma}_{zz} + \sigma_{zz}^{em}) = 0 \tag{33}$$

$$\gamma n_x + \hat{g}_x + \partial_z \Pi_{zx} = 0 \tag{34}$$

$$\gamma n_z - \frac{\partial F}{\partial n_z} + \hat{g}_z + \partial_z \Pi_{zz} = 0 \tag{35}$$

The zeros at the right hand side come from the fact that we are looking at the stationary situation. All quantities are defined from Eqs. (17)–(31). Equation (33) is not of interest in the present problem and will not be considered any further. Then we have three coupled nonlinear differential equations in $u(z)$, $\theta(z)$ and the constant γ.

To solve these equations we need boundary conditions. Firstly we use the ordinary nonslip boundary condition for liquids which gives us

$$u(z = d/2) = V, \qquad u(z = -d/2) = 0 \tag{36}$$

Secondly the director is constrained to be perpendicular to the plates i.e.

$$\theta(z = \pm d/2) = 0 \tag{37}$$

This is achieved experimentally by painting a thin lecithin film on the plates before putting the nematic between them.

Equation (32) can easily be integrated to give

$$u'g(\theta) + \tfrac{1}{2}\varepsilon_a E_{\mathrm{rms}}^2 \sin\theta\cos\theta = a \tag{38}$$

where a is an integration constant to be determined later, and $g(\theta)$ is defined by

$$\begin{aligned}2g(\theta) &= 2\alpha_1\sin^2\theta\cos^2\theta + (\alpha_5 - \alpha_2)\cos^2\theta + (\alpha_6 + \alpha_3)\sin^2\theta + \alpha_4 \\ &= 2\eta_1\sin^2\theta + 2\eta_2\cos^2\theta + 2\alpha_1\cos^2\theta\sin^2\theta\end{aligned} \tag{39}$$

We have introduced new viscosity coefficients (Miesowicz)

$$\begin{aligned}2\eta_1 &= \alpha_3 + \alpha_4 + \alpha_6 \\ 2\eta_2 &= \alpha_4 + \alpha_5 - \alpha_2\end{aligned} \tag{40}$$

Furthermore, by eliminating γ between Eqs. (34) and (35) it is a straightforward matter to arrive at the following equation

$$\theta''(k_{11}\sin^2\theta + k_{33}\cos^2\theta) + \theta'^2(k_{11} - k_{33})\sin\theta\cos\theta + \tfrac{1}{2}u'(-\gamma_1 + \gamma_2\cos 2\theta) - \varepsilon_a E_{\mathrm{rms}}^2\cos\theta\sin\theta = 0 \tag{41}$$

In this expression we introduce the following

$$f(\theta) = k_{11}\sin^2\theta + k_{33}\cos^2\theta \tag{42}$$

$$\cos 2\theta_0 = \gamma_1/\gamma_2 \tag{43}$$

If we also substitute u' as given by Eq. (38) we obtain

$$2f(\theta)\theta'' + \frac{df}{d\theta}\theta'^2 + \frac{\cos 2\theta - \cos 2\theta_0}{g(\theta)}\gamma_2(a - \tfrac{1}{3}\varepsilon_a E_{\mathrm{rms}}^2\cos\theta\sin\theta) - 2\varepsilon_a E_{\mathrm{rms}}^2\cos\theta\sin\theta = 0 \tag{44}$$

Finally we introduce a dimensionless coordinate defined by $\zeta = 2z/d$ which gives

$$2f(\theta)\frac{d^2\theta}{d\zeta^2} + \frac{df}{d\theta}\left(\frac{d\theta}{d\zeta}\right)^2 + \frac{\cos 2\theta - \cos 2\theta_0}{g(\theta)}\gamma_2(a_0 - \kappa_0\cos\theta\sin\theta) - 4\kappa_0\cos\theta\sin\theta = 0 \tag{45}$$

and Eq. (38) becomes

$$\frac{du}{d\zeta} = \frac{2}{d}\frac{a_0 - \kappa_0\cos\theta\sin\theta}{g(\theta)} \tag{46}$$

The quantities a_0 and κ_0 are given by $a_0 = ad^2/4$ and $\kappa_0 = \varepsilon_a E_{\mathrm{rms}}^2 d^2/8 = \varepsilon_a U_{\mathrm{rms}}^2/8$ respectively. Equations (45) and (46) have to be solved subject to the boundary conditions

$$\theta(\pm 1) = 0, \qquad u(-1) = 0, \qquad u(1) = V \tag{47}$$

if we want an exact description of the stationary flow. However, as can be seen from the form of the expressions, this is not easily done, even if Eq (45) can be integrated once to give

$$f(\theta)\left(\frac{d\theta}{d\zeta}\right)^2 = -\int_{\theta_2}^{\theta} G(\varphi)d\varphi \tag{48}$$

where $G(\theta)$ represents the two last terms in (45) and $0 \leq \theta \leq \theta_2 \leq \theta_0$. The integral in this formula can be evaluated analytically but little is gained by that, since the result is extremely complicated.

4 AN APPROXIMATE SOLUTION OF THE PROBLEM

In the introduction we gave a relation between the experimentally measured ring number and the mean optic anisotropy

$$m\lambda_0/d = \langle n_e - n_0 \rangle \tag{49}$$

where $\langle n_e - n_0 \rangle$ is given by

$$\langle n_e - n_0 \rangle = \int_0^1 (n_e(\zeta) - n_0)d\zeta \tag{50}$$

λ_0 is the wavelength of the incident light and d is the thickness of the sample. The integrand in Eq. (50) is given by

$$n_e - n_0 = n_{\parallel}\left\{\frac{1}{[1 - (1 - n_{\parallel}^2/n_{\perp}^2)\sin^2\theta]^{1/2}} - 1\right\} \tag{51}$$

In this expression $n_{\perp}$ and $n_{\parallel}$ are the refraction indices perpendicular and parallell to the molecule respectively and θ is the tilt angle defined in Figure 1. If θ is small, $\theta \ll 1$, this can be expanded into

$$n_e - n_0 = \tfrac{1}{2}n_{\parallel}\left(1 - \frac{n_{\parallel}^2}{n_{\perp}^2}\right)(\theta^2 - \tfrac{1}{3}\theta^4 \cdots) \tag{52}$$

In the experiment described in the introduction the shear velocity and the field are adjusted so that θ always is a small angle. With this in mind we make the following ansatz for θ:

$$\theta = \theta_2(1 - \zeta^2)(1 + \varepsilon\zeta^2) \tag{53}$$

Here θ_2 is the maximum tilt angle in the mittle of the sample and ε is a small parameter. The ansatz is motivated by the fact that when all nonlinear terms in the differential Eq. (45) are deleted, the solution of the resulting equation involves a hyperbolic cosine which can be extremely well approximated by an expression of the form Eq. (53). In the following analysis we use θ as an expansion parameter and retain terms of first and second order. First of all by using $\theta \simeq \theta_2(1 - \zeta^2)/(1 - \varepsilon\zeta^2)$ the first order derivative will be given by

$$\left(\frac{d\theta}{d\zeta}\right)^2 = \frac{4\theta_2^2}{(1 - \varepsilon)^2} - \theta\frac{4\theta_2}{(1 - \varepsilon)^2}(1 + 3\varepsilon) - \cdots \tag{54}$$

and the second order derivative by

$$\frac{d^2\theta}{d\zeta^2} = 2\theta_2\left(-1 - 5\varepsilon + 6\varepsilon\frac{\theta}{\theta_2}\cdots\right) \tag{55}$$

If this inserted into Eq. (45) and the other terms are expanded the result will be as follows. The zeroth order term gives

$$\theta_2(1 + 5\varepsilon) = \frac{a_0(\gamma_2 - \gamma_1)}{4\eta_2 k_{33}} \equiv \beta \tag{56}$$

i.e.

$$\theta_2 = \beta/(1 + 5\varepsilon) \tag{57}$$

and the first order term gives

$$\varepsilon = \tfrac{1}{3}K\beta^2 + \frac{\kappa_0}{24k_{33}}\left(\frac{\gamma_2 - \gamma_1}{\eta_2} + 4\right) \tag{58}$$

where $K = (k_{33} - k_{11})/k_{33}$.

Using Eq. (53) it is easy to calculate the integral Eq. (50) and the result is, keeping terms up to first order in ε,

$$\langle n_e - n_0\rangle = \frac{n_{\parallel}}{2}\left(1 - \frac{n_{\parallel}^2}{n_{\perp}^2}\right)\{\tfrac{8}{15}\theta_2^2(1 + \tfrac{2}{7}\varepsilon) - \tfrac{128}{945}\theta_2^4(1 + \tfrac{4}{11}\varepsilon)\} \tag{59}$$

In this expression we insert Eq. (57) and (58) with the shorthand notations $\kappa_0/(24k_{33}) = \delta$, $(\gamma_2 - \gamma_1)/\eta_2 = -\mu$ and the resulting expression is

$$\langle n_e - n_0\rangle = \frac{n_{\parallel}}{2}\left(1 - \frac{n_{\parallel}^2}{n_{\perp}^2}\right)\{\tfrac{8}{15}\beta^2[1 - \tfrac{68}{7}\delta(4 - \mu)] + \tfrac{32}{315}\beta^4[-17K - \tfrac{4}{3} + \tfrac{288}{11}\delta(4 - \mu)]\} \tag{60}$$

The final step is to relate β to the experimental situation. To arrive at a useable expression we start from Eq. (40) for the velocity gradient and expand the right hand side in θ up to first order which gives

$$\frac{du}{d\zeta} = \frac{2}{d}\left(\frac{a_0}{\eta_2} - \frac{\kappa_0}{\eta_2}\theta(\zeta)\right) \tag{61}$$

This is a separable differential equation and is integrated as follows

$$\int_0^V du = \frac{2}{d}\int_{-1}^{1}\left\{\frac{a_0}{\eta_2} - \frac{\kappa_0}{\eta_2}\theta_2(1 - \zeta^2)(1 + \varepsilon\zeta^2)\right\}d\zeta \tag{62}$$

with the result (we use $\varepsilon \ll 1$)

$$Vd = 4a_0/\eta_2 - \tfrac{8}{3}\theta_2\kappa_0/\eta_2 \simeq 4a_0/\eta_2 - \tfrac{8}{3}\beta\kappa_0/\eta_2 \tag{63}$$

This gives $a_0/\eta_2 = Vd/4 + 2\kappa_0\beta/(3\eta_2)$ which is inserted into Eq. (56) to give

$$\beta = -\frac{\mu\eta_2}{16k_{33}}\frac{Vd}{1 + \frac{1}{6}\delta\mu} \tag{64}$$

Finally this is inserted into Eq. (60) and we get

$$m/d = \langle n_e - n_0 \rangle/\lambda_0 = a_2(Vd)^2 + a_4(Vd)^4 \tag{65}$$

where

$$a_2 = \frac{n_\parallel}{2\lambda_0}\left(1 - \frac{n_\parallel^2}{n_\perp^2}\right)\frac{8}{15}\left(\frac{\mu\eta_2}{16k_{33}}\right)^2 \frac{1}{(1 + \delta\mu/\delta)^2}(1 - 68\delta(4 - \mu)/7)$$
$$a_4 = \frac{n_\parallel}{2\lambda_0}\left(1 - \frac{n_\parallel^2}{n_\perp^2}\right)\frac{16}{315}\left(\frac{\mu\eta_2}{16k_{33}}\right)^4 \frac{1}{(1 + \delta\mu/\delta)^4}\left(-17K - \tfrac{4}{3} + \tfrac{288}{11}\delta(4 - \mu)\right) \tag{66}$$

By plotting $m/(d(Vd)^2)$ as a function of $(Vd)^2$ i.e.

$$m/(d(Vd)^2) = a_2 + a_4(Vd)^4 \tag{67}$$

one can determine a_2 and a_4 as functions of the external parameter $\kappa_0 = 24k_{33}\delta$. The result will look as in Figures 2 and 3. For zero field (i.e. $\kappa_0 = 0$) one can find $\mu\eta_2/k_{33}$ and K from Eq. (66). By a fitting procedure it is then possible to find δ and μ separately. In this way it is easy to determine $k_{11}, k_{33}, \gamma_1 - \gamma_2$ and η_2. Finally γ_1/γ_2 can be determined when the nematic is sheared with high velocity.

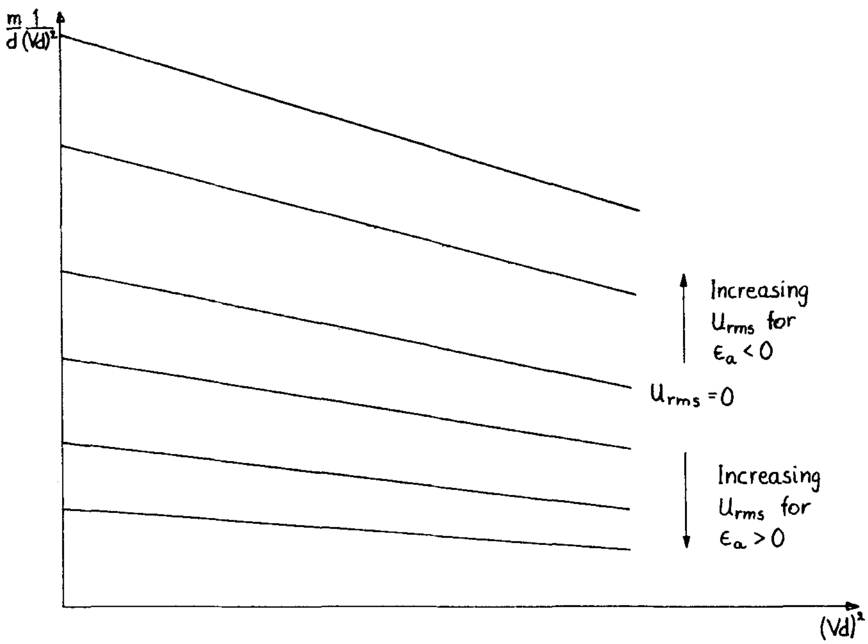

FIGURE 2 Plots of $m/((Vd)^2d) = a_2 + a_4(Vd)^2$ for different voltages and $\varepsilon_a > 0$ and $\varepsilon_a < 0$.

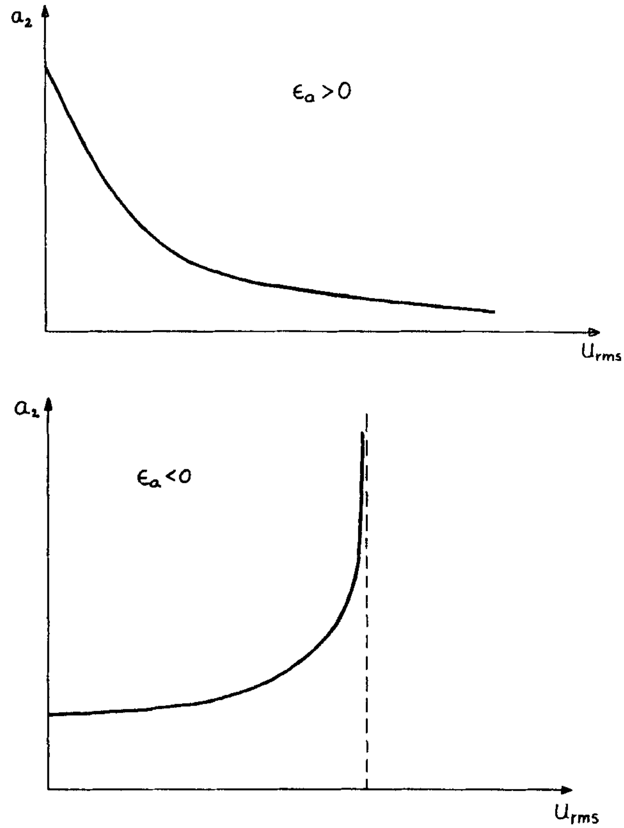

FIGURE 3 The function $a_2(U_{rms})$ for $\varepsilon_a > 0$ and $\varepsilon_a < 0$.

5 DISCUSSION

Two comments have to be made. The first one concerns what will happen if a nematic with negative dielectric anisotropy is used and the voltage is increased. The second one concerns the limits within which the approximation is valid.

As can be seen from Eq. (66) both a_2 and a_4 diverge as the expression $1 + \delta\mu/6$ approaches zero. This can only happen for $\varepsilon_a < 0$ because then $\delta < 0$. Since we start with $\delta = 0$ we must have the criterion

$$1 + \delta\mu/6 > 0 \tag{68}$$

Inserting the expressions for δ and μ this leads to

$$U_{\text{rms}}^2 < \frac{1152k_{33}}{|\varepsilon_a|}\frac{\eta_2}{\gamma_1 - \gamma_2} \tag{69}$$

For MBBA this gives a critical voltage of about 83 V which is below the critical voltage (~ 100 V) for transition from the stable to the unstable dielectric regime.

The smallness of the parameter ε defined in Eqs. (53) and (58) puts a limit to the applied voltage for both $\varepsilon_a > 0$ and $\varepsilon_a < 0$. If we for the moment forget the first term in Eq. (58) we can express $U_{\rm rms}$ in the other parameters,

$$U_{\rm rms}^2 = \frac{8k_{33}\varepsilon}{|\varepsilon_a|}\,\frac{24}{4-(\gamma_1-\gamma_2/\eta_2)} \tag{70}$$

For $K15$ this gives a voltage

$$U_{\rm rms} = 1.96\varepsilon^{1/2} \tag{71}$$

It is then the required smallness of ε that limits $U_{\rm rms}$. How small must ε be to be considered small? This is somewhat arbitrary, but we think that an upper limit for ε may be as high as 0.5. For $K15$ this corresponds to $U_{\rm rms} = 1.4$ V. In practice it seems that the experiment can be carried on to even higher $U_{\rm rms}$ without any problems.[24] To understand this we have started to make a numerical solution of the full non-linear equations. In this way we hope to see when and how the approximation deviates significantly from an exact solution.

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